Int. J. Solids Structures, 1972, Vol. 8, pp. 895 to 906. Pergamon Press. Printed in Great Britain

HIGH FREQUENCY VIBRATIONS OF PIEZOELECTRIC CRYSTAL PLATES

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Abstract—Two-dimensional equations of motion of piezoelectric crystal plates are obtained by retaining early terms of power series expansions of the mechanical displacement and electric potential in a variational principle for the three-dimensional equations of piezoelectricity.

INTRODUCTION

There have been two previous formulations of two-dimensional equations of motion of piezoelectric crystal plates applicable to frequencies as high as those of the fundamental thickness-shear modes. In the first [1], the reduction of the classical equations of piezoelectricity from three to two dimensions was based on an approximation involving the early terms of series expansions of the mechanical displacement and electric potential in powers of the thickness coordinate of the plate. Only flexure, thickness–shear and thickness–twist modes were taken into account. The equations were extended, subsequently [2], to include the face–extension and face–shear modes (the "contour" modes) in a treatment based on power series expansions of the mechanical displacement and electric displacement. In the present paper, equations of somewhat simpler form, including the flexure, thickness–shear, thickness–twist and contour modes, are obtained by means of power series expansions of the mechanical displacement and electric potential. The derivation employed is a revision (to introduce a variational principle) and an extension (to include the electric field and the thickness modes) of one devised by Cauchy [3] to deduce the classical equations of low frequency vibrations of anisotropic plates (flexure and contour modes only) from the three-dimensional equations of elasticity. The two-dimensional equations are derived in condensed form for the triclinic crystal, but they are displayed in detail for the case of the rotated-Y-cuts of quartz—the crystal cuts most widely used for resonators at the present time. Shear correction factors, of the type employed by Bresse [4] and Timoshenko [5] in the theory of beams, are introduced and their values are determined by equating the thickness–shear frequencies obtained from the two- and three-dimensional equations [1].

THREE-DIMENSIONAL EQUATIONS

The three-dimensional equations of piezoelectricity, from which the two-dimensional equations are to be deduced, follow.

The field equations are

$$T_{ij,i} = \rho \ddot{u}_j, \qquad D_{i,i} = 0, \tag{1}$$

where the T_{ij}, u_j and D_i are the components of stress, mechanical displacement and electric displacement, respectively, and ρ is the mass density.

The constitutive equations are

$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \qquad D_i = e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \tag{2}$$

where the c_{ijkl}, e_{kij} and ε_{ij} are the components of elastic stiffness, piezoelectric strain constant and dielectric permittivity, respectively, and the S_{ij} and E_i are the components of strain and electric field—expressed in terms of the u_i and the electric potential φ by

$$S_{ij} = \tfrac{1}{2}(u_{j,i}+u_{i,j}), \qquad E_i = -\varphi_{,i}. \tag{3}$$

Combining (1)–(3), we have the equations governing the u_i and φ:

$$c_{ijkl}u_{k,li} + e_{kij}\varphi_{,ki} = \rho\ddot{u}_j, \qquad e_{kij}u_{i,jk} - \varepsilon_{ij}\varphi_{,ij} = 0. \tag{4}$$

These equations may be derived from the following variational principle: in a region V bounded by a surface S, with outward normal $\mathbf{n}$,

$$\delta\int_{t_0}^{t_1} \mathrm{d}t \int_V (K-H)\,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S (t_j\,\delta u_j + \sigma\,\delta\varphi)\,\mathrm{d}S = 0 \tag{5}$$

for independent variations of the u_j and φ between fixed limits at times t_0 and t_1. In (5), K is the kinetic energy density:

$$K = \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i$$

and H is the electric enthalpy density (the energy density less E_iD_i):

$$H = \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl} - \tfrac{1}{2}\varepsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk}$$

so that

$$T_{ij} = \partial H/\partial S_{ij}, \qquad D_i = -\partial H/\partial E_i.$$

Also, in (5) t_j is the surface traction and σ is the surface charge. We have

$$\delta\int_{t_0}^{t_1} K\,\mathrm{d}t = \int_{t_0}^{t_1} \rho\dot{u}_j\,\delta\dot{u}_j\,\mathrm{d}t = [\rho\dot{u}_j\,\delta u_j]_{t_0}^{t_1} - \int_{t_0}^{t_1} \rho\ddot{u}_j\,\delta u_j\,\mathrm{d}t = -\int_{t_0}^{t_1} \rho\ddot{u}_j\,\delta u_j\,\mathrm{d}t,$$

$$\begin{aligned}\delta\int_V H\,\mathrm{d}V &= \int_V [(\partial H/\partial S_{ij})\,\delta S_{ij} + (\partial H/\partial E_i)\,\delta E_i]\,\mathrm{d}V = \int_V (T_{ij}\,\delta u_{j,i} + D_i\,\delta\varphi_{,i})\,\mathrm{d}V \\ &= \int_V [(T_{ij}\,\delta u_j)_{,i} - T_{ij,i}\,\delta u_j + (D_i\,\delta\varphi)_{,i} - D_{i,i}\,\delta\varphi]\,\mathrm{d}V \\ &= \int_S n_i(T_{ij}\,\delta u_j + D_i\,\delta\varphi)\,\mathrm{d}S - \int_V (T_{ij,i}\,\delta u_j + D_{i,i}\,\delta\varphi)\,\mathrm{d}V.\end{aligned}$$

Hence, (5) becomes

$$\int_{t_0}^{t_1} \mathrm{d}t \int_V [(T_{ij,i} - \rho\ddot{u}_j)\,\delta u_j + D_{i,i}\,\delta\varphi]\,\mathrm{d}V + \int_{t_0}^{t_1} \mathrm{d}t \int_S [(t_j - n_iT_{ij})\,\delta u_j + (\sigma - n_iD_i)\,\delta\varphi]\,\mathrm{d}S = 0,$$

from which follow the natural boundary conditions

$$n_iT_{ij} = t_j, \qquad n_iD_i = \sigma \quad \text{on } S \tag{6}$$

or, alternatively, specification of surface displacement $\bar{u}$ and surface potential $\bar{\varphi}$:

$$u_i = \bar{u}_i, \qquad \varphi = \bar{\varphi} \quad \text{on } S \tag{7}$$

and the field equations (1) as Euler equations which, with (2) and (3), produce the equations (4) on the u_i and φ.

SERIES OF TWO-DIMENSIONAL EQUATIONS

The plate is referred to rectangular coordinates x_i with the faces, of area A, at $x_2 = \pm b$ and with x_1 and x_3 the coordinates of the middle plane which intersects the right cylindrical or prismatic boundary of the plate in a curve C. We assume that the mechanical displacement and electric potential can be approximated by power series in x_2:

$$u_i = \sum_n x_2^n u_i^{(n)}, \qquad \varphi = \sum_n x_2^n \varphi^{(n)}, \tag{8}$$

where $u_i^{(n)}$ and $\varphi^{(n)}$, $n = 1, 2 \dots k$, are functions of x_1, x_3 and t only. Then, from (8) and (3),

$$S_{ij} = \sum_n x_2^n S_{ij}^{(n)}, \qquad E_i = \sum_n x_2^n E_i^{(n)}, \tag{9}$$

where

$$S_{ij}^{(n)} = \tfrac{1}{2}[u_{j,i}^{(n)} + u_{i,j}^{(n)} + (n+1)(\delta_{i2}u_j^{(n+1)} + \delta_{2j}u_i^{(n+1)})], \qquad E_i^{(n)} = -\varphi_{,i}^{(n)} - (n+1)\delta_{2i}\varphi^{(n+1)}. \tag{10}$$

We have now to reduce the variational principle (5) to two dimensions. Consider, first, the surface integral and separate it into integrals over the faces and the cylindrical boundary:

$$\begin{aligned}\int_S (t_j\,\delta u_j + \sigma\,\delta\varphi)\,\mathrm{d}S &= \int_S n_i(T_{ij}\,\delta u_j + D_i\,\delta\varphi)\,\mathrm{d}S \\ &= \sum_n \int_A [x_2^n(T_{2j}\,\delta u_j^{(n)} + D_2\,\delta\varphi^{(n)}]_{-b}^{b}\,\mathrm{d}A \\ &\quad + \sum_n \oint_C \int_{-b}^{b} x_2^n n_a(T_{aj}\,\delta u_j^{(n)} + D_a\,\delta\varphi^{(n)})\,\mathrm{d}x_2\,\mathrm{d}s,\end{aligned}$$

where the index a ranges over 1 and 3 only and s is the coordinate measured along the curve C. Define face-tractions $T_j^{(n)}$, face-charges $D^{(n)}$, edge-tractions $t^{(n)}$ and edge-charges $d^{(n)}$ according to

$$T_j^{(n)} = B_n^{-1}[x_2^n T_{2j}]_{-b}^{b}, \qquad D^{(n)} = B_n^{-1}[x_2^n D_2]_{-b}^{b},$$

$$t_j^{(n)} = B_n^{-1}\int_{-b}^{b} x_2^n(n_a T_{aj})_C\,\mathrm{d}x_2, \qquad d^{(n)} = B_n^{-1}\int_{-b}^{b} x_2^n(n_a D_a)_C\,\mathrm{d}x_2,$$

where

$$B_n = 2b^{2n+1}/(2n+1).$$

Also, define

$$\bar{K} = \int_{-b}^{b} K\,\mathrm{d}x_2, \qquad \bar{H} = \int_{-b}^{b} H\,\mathrm{d}x_2.$$

Then the variational principle (5) becomes the two-dimensional one: for independent variations $\delta u_j^{(n)}$ and $\delta\varphi^{(n)}$ between fixed limits at t_0 and t_1:

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \int_A (\bar{K}-\bar{H})\,\mathrm{d}A + \int_{t_0}^{t_1} \mathrm{d}t \int_A \sum_n B_n(T_j^{(n)}\,\delta u_j^{(n)} + D^{(n)}\,\delta\varphi^{(n)})\,\mathrm{d}A$$
$$+ \int_{t_0}^{t_1} \mathrm{d}t \oint_C \sum_n B_n(t_j^{(n)}\,\delta u_j^{(n)} + d^{(n)}\,\delta\varphi^{(n)})\,\mathrm{d}s = 0. \tag{11}$$

For the kinetic energy density $\bar{K}$, we have

$$\bar{K} = \tfrac{1}{2}\rho \int_{-b}^{b} \dot{u}_j\dot{u}_j\,\mathrm{d}x_2 = \tfrac{1}{2}\rho \int_{-b}^{b} \left(\sum_m x_2^m \dot{u}_j^{(m)}\right)\left(\sum_n x_2^n \dot{u}_j^{(n)}\right)\mathrm{d}x_2 = \tfrac{1}{2}\rho \sum_m \sum_n B_{mn}\dot{u}_j^{(m)}\dot{u}_j^{(n)},$$

where

$$B_{mn} = 2b^{m+n+1}/(m+n+1), \qquad m+n \text{ even}$$

and $B_{mn} = 0$ for $m+n$ odd. Further,

$$\int_{t_0}^{t_1} \delta\bar{K}\,\mathrm{d}t = \rho \sum_m \sum_n B_{mn}[\dot{u}_j^{(m)}\,\delta u_j^{(n)}]_{t_0}^{t_1} - \rho \int_{t_0}^{t_1} \sum_m \sum_n B_{mn}\ddot{u}_j^{(m)}\,\delta u_j^{(n)}\,\mathrm{d}t$$

or

$$\int_{t_0}^{t_1} \delta\bar{K}\,\mathrm{d}t = -\rho \int_{t_0}^{t_1} \sum_m \sum_n B_{mn}\ddot{u}_j^{(m)}\,\delta u_j^{(n)}\,\mathrm{d}t. \tag{12}$$

As for the electric enthalpy density $\bar{H}$, we have

$$\delta\bar{H} = \int_{-b}^{b} \delta H\,\mathrm{d}x_2 = \int_{-b}^{b} (T_{ij}\,\delta S_{ij} - D_i\,\delta E_i)\,\mathrm{d}x_2$$

or, with (9),

$$\delta\bar{H} = \int_{-b}^{b} \sum_n x_2^n(T_{ij}\,\delta S_{ij}^{(n)} - D_i\,\delta E_i^{(n)})\,\mathrm{d}x_2 = \sum_n (T_{ij}^{(n)}\,\delta S_{ij}^{(n)} - D_i^{(n)}\,\delta E_i^{(n)}), \tag{13}$$

where $T_{ij}^{(n)}$ and $D_i^{(n)}$ are the components of stress and electric displacement of order n:

$$T_{ij}^{(n)} = \int_{-b}^{b} x_2^n T_{ij}\,\mathrm{d}x_2, \qquad D_i^{(n)} = \int_{-b}^{b} x_2^n D_i\,\mathrm{d}x_2.$$

Upon substituting (10) into (13), we have

$$\begin{aligned}\delta\bar{H} &= \sum_n \{T_{ij}^{(n)}[\delta u_{j,i}^{(n)} + (n+1)\delta_{i2}\,\delta u_j^{(n+1)}] + D_i^{(n)}[\delta\varphi_{,i}^{(n)} + (n+1)\,\delta\varphi^{(n+1)}]\} \\ &= \sum_n T_{ij}^{(n)}\,\delta u_j^{(n)} + D_i^{(n)}\,\delta\varphi^{(n)})_{,i} - \sum_n [(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)})\,\delta u_j^{(n)} + (D_{i,i}^{(n)} - nD_2^{(n-1)})\,\delta\varphi^{(n)}],\end{aligned}$$

whence

$$\int_A \delta \bar{H}\, \mathrm{d}A = \oint_C \sum_n n_a (T_{aj}^{(n)}\, \delta u_j^{(n)} + D_a^{(n)}\, \delta\varphi^{(n)})\, \mathrm{d}s$$
$$- \int_A \sum_n [(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)})\, \delta u_j^{(n)} + (D_{i,i}^{(n)} - nD_2^{(n-1)})\, \delta\varphi^{(n)}]\, \mathrm{d}A. \tag{14}$$

Inserting (12) and (14) in (11), we arrive at the series form of the two-dimensional variational equation:

$$\sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A \left(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_n T_j^{(n)} - \rho \sum_m B_{mn} \ddot{u}_j^{(m)} \right) \delta u_j^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \int_A (D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)})\, \delta\varphi^{(n)}\, \mathrm{d}A$$
$$+ \sum_n \int_{t_0}^{t_1} \mathrm{d}t \oint_C [(B_n t_j^{(n)} - n_a T_{aj}^{(n)})\, \delta u_j^{(n)} + (B_n d^{(n)} - n_a D_a^{(n)})\, \delta\varphi^{(n)}]\, \mathrm{d}s = 0,$$

from which follow the field equations, in A,

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_n T_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)}, \qquad D_{i,i}^{(n)} - nD_2^{(n-1)} + B_n D^{(n)} = 0$$

and boundary conditions

$$n_a T_{aj}^{(n)} = B_n t_j^{(n)}, \qquad n_a D_a^{(n)} = B_n d^{(n)} \quad \text{on } C.$$

Alternative boundary conditions are prescribed displacements $\bar{u}_j^{(n)}$ and prescribed potentials $\bar{\varphi}^{(n)}$ on C:

$$u_j^{(n)} = \bar{u}_j^{(n)}, \qquad \varphi^{(n)} = \bar{\varphi}^{(n)} \quad \text{on } C.$$

The constitutive equations of order n are obtained as follows:

$$T_{ij}^{(n)} = \int_{-b}^{b} x_2^n T_{ij}\, \mathrm{d}x_2 = \int_{-b}^{b} x_2^n (c_{ijkl} S_{kl} - e_{kij} E_k)\, \mathrm{d}x_2 = \int_{-b}^{b} \sum_m x_2^m x_2^n (c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)})\, \mathrm{d}x_2,$$

$$D_i^{(n)} = \int_{-b}^{b} x_2^n D_i\, \mathrm{d}x_2 = \int_{-b}^{b} x_2^n (e_{ijk} S_{jk} + \varepsilon_{ij} E_j)\, \mathrm{d}x_2 = \int_{-b}^{b} \sum_m x_2^m x_2^n (e_{ijk} S_{jk}^{(m)} + \varepsilon_{ij} E_j^{(m)})\, \mathrm{d}x_2,$$

whence,

$$T_{ij}^{(n)} = \sum_m B_{mn} (c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)}), \qquad D_i^{(n)} = \sum_m B_{mn} (e_{ijk} S_{jk}^{(m)} + \varepsilon_{ij} E_j^{(m)}). \tag{15}$$

TRUNCATION OF SERIES AND ADJUSTMENT

The process of truncation of the series and adjustment of the remaining terms begins with the discard of the strains and electric fields of order higher than the first, leaving $S_{ij}^{(0)}$, $S_{ij}^{(1)}$, $E_i^{(0)}$, $E_i^{(1)}$ which contain the zero, first and second order mechanical displacements, $u_j^{(0)}$, $u_j^{(1)}$, $u_j^{(2)}$ and electric potentials, $\varphi^{(0)}$, $\varphi^{(1)}$, $\varphi^{(2)}$, some of which will be eliminated subsequently. However, at this stage the constitutive equations (15) reduce to

$$T_{ij}^{(0)} = 2b(c_{ijkl}S_{kl}^{(0)} - e_{kij}E_k^{(0)}), \qquad D_i^{(0)} = 2b(e_{ijk}S_{jk}^{(0)} + \varepsilon_{ij}E_j^{(0)}), \tag{16}$$

$$T_{ij}^{(1)} = \tfrac{2}{3}b^3(c_{ijkl}S_{kl}^{(1)} - e_{kij}E_k^{(1)}), \qquad D_i^{(1)} = \tfrac{2}{3}b^3(e_{ijk}S_{jk}^{(1)} + \varepsilon_{ij}E_j^{(1)}) \tag{17}$$

and these are derivable from the electric enthalpy density

$$\bar{H} = b(c_{ijkl}S_{ij}^{(0)}S_{kl}^{(0)} - \varepsilon_{ij}E_i^{(0)}E_j^{(0)} - 2e_{ijk}E_i^{(0)}S_{jk}^{(0)}) + \tfrac{1}{3}b^3(c_{ijkl}S_{ij}^{(1)}S_{kl}^{(1)} - \varepsilon_{ij}E_i^{(1)}E_j^{(1)} - 2e_{ijk}E_i^{(1)}S_{jk}^{(1)})$$

according to

$$T_{ij}^{(n)} = \partial\bar{H}/\partial S_{ij}^{(n)}, \qquad D_i^{(n)} = -\partial\bar{H}/\partial E_i^{(n)}, \qquad n = 0, 1.$$

Next, following Cauchy [3], we neglect the velocity $\dot{u}_2^{(1)}$ in the kinetic energy density, and provide for free development of the strain $S_{22}^{(0)}(=u_2^{(1)})$ by setting $T_{22}^{(0)} = 0$ in (16). Thus,

$$T_{22}^{(0)} = 2b(c_{22kl}S_{kl}^{(0)} - e_{k22}E_k^{(0)}) = 0. \tag{18}$$

Add $c_{2222}S_{22}^{(0)}$ to each side of (18) and divide by c_{2222}:

$$S_{22}^{(0)} = -c_{22kl}S_{kl}^{(0)}/c_{2222} + S_{22}^{(0)} + e_{k22}E_k^{(0)}/c_{2222}. \tag{19}$$

Now, add and subtract $c_{ij22}S_{22}^{(0)}$ to the expression for $T_{ij}^{(0)}$ in (16):

$$(2b)^{-1}T_{ij}^{(0)} = (c_{ijkl}S_{kl}^{(0)} - c_{ij22}S_{22}^{(0)}) + c_{ij22}S_{22}^{(0)} - e_{kij}E_k^{(0)}. \tag{20}$$

Finally, substitute the expression for $S_{22}^{(0)}$, given in (19), for the $S_{22}^{(0)}$ outside the parentheses in (20), to obtain

$$T_{ij}^{(0)} = 2b(\bar{c}_{ijkl}S_{kl}^{(0)} - \bar{e}_{kij}E_k^{(0)}), \tag{21}$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - c_{ij22}c_{22kl}/c_{2222}, \qquad \bar{e}_{kij} = e_{kij} - e_{k22}c_{ij22}/c_{2222}.$$

Note that, in (21), $T_{22}^{(0)}$ is now zero and $S_{22}^{(0)}$ $(=u_2^{(1)})$ is no longer present.

The first order terms are treated similarly except that all three velocities $\dot{u}_j^{(2)}$ are neglected in the kinetic energy density and free development of the three strains $S_{2j}^{(1)}$ is accommodated by setting $T_{2j}^{(1)} = 0$ in (17). When so many components of stress are to be set equal to zero, it is simpler to start with expressions for the strains in terms of the stresses. To do this, first define elastic compliances, s_{ijkl}, and piezoelectric strain constants, d_{ijk}, according to [6]

$$s_{ijmn}c_{ijkl} = I_{mnkl}, \qquad d_{kmn} = s_{ijmn}e_{kij},$$

where I_{mnkl} is the fourth rank unit tensor. Then multiply the formula for $T_{ij}^{(1)}$, in (17), by s_{ijmn} to obtain

$$s_{ijmn}T_{ij}^{(1)} = \tfrac{2}{3}b^3(S_{mn}^{(1)} - d_{kmn}E_k^{(1)}). \tag{22}$$

In (22), set $T_{2j}^{(1)} = 0$, $\varphi^{(2)} = 0$, so that we may write

$$s_{abcd}T_{cd}^{(1)} = \tfrac{2}{3}b^3(S_{ab}^{(1)} - d_{cab}E_c^{(1)}), \tag{23}$$

where the subscripts a, b, c, d range over 1 and 3 only. Now, solve the three equations (23) for the three independent $T_{cd}^{(1)}$:

$$T_{ab}^{(1)} = \tfrac{2}{3}b^3A_{abcd}(S_{cd}^{(1)} - d_{ecd}E_e^{(1)})/|s_{abcd}|, \tag{24}$$

where the determinant $|s_{abcd}|$ is given by

$$|s_{abcd}| = \begin{vmatrix} s_{1111} & s_{3311} & s_{1311} \\ s_{1133} & s_{3333} & s_{1333} \\ s_{1113} & s_{3313} & s_{1313} \end{vmatrix}$$

and A_{abcd} is the cofactor of s_{abcd} in $|s_{abcd}|$. The expression (24) may be written in the form

$$T_{ab}^{(1)} = \tfrac{2}{3}b^3(c_{abcd}^{(1)}S_{cd}^{(1)} - e_{cab}^{(1)}E_c^{(1)}), \tag{25}$$

where

$$c_{abcd}^{(1)} = A_{abcd}/|s_{abcd}|, \qquad e_{cab}^{(1)} = d_{cde}c_{abde}^{(1)}.$$

The constants $c_{abcd}^{(1)}$ are Voigt's [7] constants γ_{pq}, $p, q = 1, 3, 5$.

At this stage, an electric enthalpy density that produces the stresses $T_{ij}^{(0)}$ in (21) and $T_{ab}^{(1)}$ in (24) is

$$\begin{aligned} \bar{H} &= b(\bar{c}_{ijkl}S_{ij}^{(0)}S_{kl}^{(0)} - \varepsilon_{ij}E_i^{(0)}E_j^{(0)} - 2\bar{e}_{kij}E_k^{(0)}S_{ij}^{(0)}) \\ &\quad + \tfrac{1}{3}b^3(c_{abcd}^{(1)}S_{ab}^{(1)}S_{cd}^{(1)} - \varepsilon_{ab}E_a^{(1)}E_b^{(1)} - 2e_{abc}^{(1)}E_a^{(1)}S_{bc}^{(1)}) \end{aligned}$$

and this form also fixes $D_i^{(0)}$ and $D_a^{(1)}$ through

$$D_i^{(0)} = -\partial\bar{H}/\partial E_i^{(0)}, \qquad D_a^{(1)} = -\partial\bar{H}/\partial E_a^{(1)}.$$

The final adjustment is made by introducing shear correction factors analogous to those employed in beam theory by Bresse [4] and Timoshenko [5]. The thickness-shear strains $S_{21}^{(0)}$ and $S_{23}^{(0)}$ are replaced by $\kappa_1 S_{21}^{(0)}$ and $\kappa_3 S_{23}^{(0)}$ in the electric enthalpy density, where κ_1 and κ_3 are correction factors whose values may be chosen in such a way [1] that the important thickness-shear frequencies have the correct values, thus compensating, in part, for the omission of terms of higher order in the series expansions. The final form of the electric enthalpy density is, thus,

$$\begin{aligned} \bar{H} &= b(c_{ijkl}^{(0)}S_{ij}^{(0)}S_{kl}^{(0)} - \varepsilon_{ij}E_i^{(0)}E_j^{(0)} - 2e_{kij}^{(0)}E_k^{(0)}S_{ij}^{(0)}) \\ &\quad + \tfrac{1}{3}b^3(c_{abcd}^{(1)}S_{ab}^{(1)}S_{cd}^{(1)} - \varepsilon_{ab}E_a^{(1)}E_b^{(1)} - 2e_{abc}^{(1)}E_a^{(1)}S_{bc}^{(1)}) \end{aligned}$$

where

$$c_{ijkl}^{(0)} = \kappa_{i+j-2}^{\mu}\kappa_{k+l-2}^{\nu}\bar{c}_{ijkl}, \qquad e_{kij}^{(0)} = \kappa_{i+j-2}^{\mu}\bar{e}_{kij} \quad \text{(not summed)}$$

and μ and ν are the powers

$$\mu = \cos^2(ij\pi/2), \qquad \nu = \cos^2(kl\pi/2).$$

Thus, κ_{i+j-2}^{μ} (or κ_{k+l-2}^{ν}) is equal to κ_1, κ_3 or unity according as $i+j$ (or $k+l$) is 3, 5 or neither, respectively.

RECAPITULATION

The equations and variables remaining, after truncation of the series and adjustment of the terms retained, are

Kinetic energy density

$$\bar{K} = \rho b(\dot{u}_j^{(0)}\dot{u}_j^{(0)} + \tfrac{1}{3}b^2\dot{u}_a^{(1)}\dot{u}_a^{(1)}).$$

Electric enthalpy density

$$\bar{H} = b(c^{(0)}_{ijkl}S^{(0)}_{ij}S^{(0)}_{kl} - \varepsilon_{ij}E^{(0)}_i E^{(0)}_j - 2e^{(0)}_{kij}E^{(0)}_k S^{(0)}_{ij}) + \tfrac{1}{3}b^3(c^{(1)}_{abcd}S^{(1)}_{ab}S^{(1)}_{cd} - \varepsilon_{ab}E^{(1)}_a E^{(1)}_b - 2e^{(1)}_{abc}E^{(1)}_a S^{(1)}_{bc}).$$

Strain–displacement relations

$$S^{(0)}_{ij} = \tfrac{1}{2}(u^{(0)}_{j,i} + u^{(0)}_{i,j} + \delta_{i2}u^{(1)}_j + \delta_{2j}u^{(1)}_i), \qquad S^{(1)}_{ab} = \tfrac{1}{2}(u^{(1)}_{b,a} + u^{(1)}_{a,b}).$$

Electric field–potential relations

$$E^{(0)}_i = -\varphi^{(0)}_{,i} - \delta_{i2}\varphi^{(1)}, \qquad E^{(1)}_a = -\varphi^{(0)}_{,a}.$$

Constitutive equations

$$T^{(0)}_{ij} = \partial\bar{H}/\partial S^{(0)}_{ij} = 2b(c^{(0)}_{ijkl}S^{(0)}_{kl} - e^{(0)}_{kij}E^{(0)}_k), \qquad D^{(0)}_i = -\partial\bar{H}/\partial E^{(0)}_i = 2b(e^{(0)}_{ijk}S^{(0)}_{jk} + \varepsilon_{ij}E^{(0)}_j)$$

$$T^{(1)}_{ab} = \partial\bar{H}/\partial S^{(1)}_{ab} = \tfrac{2}{3}b^3(c^{(1)}_{abcd}S^{(1)}_{cd} - e^{(1)}_{cab}E^{(1)}_c), \qquad D^{(1)}_a = -\partial\bar{H}/\partial E^{(1)}_a = \tfrac{2}{3}b^3(e^{(1)}_{abc}S^{(1)}_{bc} + \varepsilon_{ab}E^{(1)}_{ab}).$$

Variational principle

$$\int_{t_0}^{t_1} \mathrm{d}t \int_A (\bar{K} - \bar{H})\,\mathrm{d}A + \int_{t_0}^{t_1} \mathrm{d}t \int_A [2b(T^{(0)}_j\,\delta u^{(0)}_j + D^{(0)}\,\delta\varphi^{(0)}) + \tfrac{2}{3}b^3(T^{(1)}_a\,\delta u^{(1)}_a + D^{(1)}\,\delta\varphi^{(1)})]\,\mathrm{d}A$$

$$+ \int_{t_0}^{t_1} \mathrm{d}t \oint_C [(2bt^{(0)}_j - n_a T^{(0)}_{aj})\,\delta u^{(0)}_j + (2bd^{(0)} - n_a D^{(0)}_a)\,\delta\varphi^{(0)}]\,\mathrm{d}s$$

$$+ \int_{t_0}^{t_1} \mathrm{d}t \oint_C [(\tfrac{2}{3}b^3 t^{(1)}_b - n_a T^{(1)}_{ab})\,\delta u^{(1)}_b + (\tfrac{2}{3}b^3 d^{(1)} - n_a D^{(1)}_a)\delta\varphi^{(1)}]\,\mathrm{d}s = 0.$$

Field equations

$$T^{(0)}_{ij,i} + 2bT^{(0)}_j = 2b\rho\ddot{u}^{(0)}_j$$

$$D^{(0)}_{i,i} + 2bD^{(0)} = 0$$

$$T^{(1)}_{ab,a} - T^{(0)}_{2b} + \tfrac{2}{3}b^3 T^{(1)}_b = \tfrac{2}{3}b^3\rho\ddot{u}^{(1)}_b$$

$$D^{(1)}_{a,a} - D^{(0)}_2 + \tfrac{2}{3}b^3 D^{(1)} = 0.$$

Edge conditions

$$n_a T^{(0)}_{aj} = 2bt^{(0)}_j \quad \text{or} \quad u^{(0)}_j = \bar{u}^{(0)}_j$$

$$n_a D^{(0)}_a = 2bd^{(0)} \quad \text{or} \quad \varphi^{(0)} = \bar{\varphi}^{(0)}$$

$$n_a T^{(1)}_{ab} = \tfrac{2}{3}b^3 t^{(1)}_b \quad \text{or} \quad u^{(1)}_b = \bar{u}^{(1)}_b$$

$$n_a D^{(1)}_a = \tfrac{2}{3}b^3 d^{(1)} \quad \text{or} \quad \varphi^{(1)} = \bar{\varphi}^{(1)}$$

Equations on $u^{(0)}_j$, $u^{(1)}_a$, $\varphi^{(0)}$, $\varphi^{(1)}$

$$c^{(0)}_{ijkl}(u^{(0)}_{k,li} + \delta_{2l}u^{(1)}_{k,i}) + e^{(0)}_{kij}\varphi^{(0)}_{,ki} + T^{(0)}_j = \rho\ddot{u}^{(0)}_j$$

$$e^{(0)}_{kij}(u^{(0)}_{j,ki} + \delta_{2i}u^{(1)}_{j,k}) - \varepsilon_{ij}\varphi^{(0)}_{,ij} + D^{(0)} = 0$$

$$c^{(1)}_{abcd}u^{(1)}_{d,ca} + e^{(1)}_{cab}\varphi^{(1)}_{,ca} - 3b^{-2}[c^{(0)}_{2bkl}(u^{(0)}_{l,k} + \delta_{2k}u^{(1)}_l) + e^{(0)}_{a2b}\varphi^{(0)}_{,a} + e^{(0)}_{22b}\varphi^{(1)}] + T^{(1)}_b = \rho\ddot{u}^{(1)}_b$$

$$e^{(1)}_{cab}u^{(1)}_{b,ca} - \varepsilon_{bc}\varphi^{(1)}_{,bc} - 3b^{-2}[e^{(2)}_{2ij}(u^{(0)}_{j,i} + \delta_{2i}u^{(1)}_j) - \varepsilon_{a2}\varphi^{(0)}_{,a} - \varepsilon_{22}\varphi^{(1)}] + D^{(1)} = 0.$$

APPLICATION TO ROTATED-Y-CUTS OF QUARTZ

After performing the summations over repeated indices in the preceding equations, it is convenient to employ the abbreviated notation in which a pair of indices, ij or kl, is replaced by a single index, p or q, ranging over 1–6 according to [6]

$$\begin{array}{lcccccc} ij \text{ or } kl = 11 & 22 & 33 & 23,32 & 31,13 & 12,21 \\ p \text{ or } q = 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

Then

$$c_{ijkl} \to c_{pq}, \qquad c^{(1)}_{ijkl} \to \gamma_{pq}, \qquad e_{kij} \to e_{kp},$$

$$s_{ijkl} \to \begin{cases} s_{pq} \text{ if } i = j, k = l, \\ \frac{1}{2}s_{pq} \text{ if } i = j, k \neq l \quad \text{or} \quad i \neq j, k = l, \\ \frac{1}{4}s_{pq} \text{ if } i \neq j, k \neq l, \end{cases}$$

$$d_{kij} \to \begin{cases} d_{kp} \text{ if } i = j, \\ \frac{1}{2}d_{kp} \text{ if } i \neq j. \end{cases}$$

For the rotated-Y-cuts of quartz, many of the constants are zero. If x_1 is the digonal axis in the plane of the plate [8]:

$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0,$$

$$s_{15} = s_{16} = s_{25} = s_{26} = s_{35} = s_{36} = s_{45} = s_{46} = 0,$$

$$e_{21} = e_{31} = e_{22} = e_{32} = e_{23} = e_{33} = e_{24} = e_{34} = e_{15} = e_{16} = 0,$$

$$d_{21} = d_{31} = d_{22} = d_{32} = d_{23} = d_{33} = d_{24} = d_{34} = d_{15} = d_{16} = 0,$$

$$\varepsilon_{12} = \varepsilon_{13} = 0.$$

The equations on $u^{(0)}_j$, $u^{(1)}_a$, $\varphi^{(0)}$, $\varphi^{(1)}$ then reduce to

$$\begin{aligned} &\bar{c}_{11}u^{(0)}_{1,11} + c_{55}u^{(0)}_{1,33} + (\kappa_1 c_{56} + \kappa_3 \bar{c}_{14})u^{(0)}_{2,13} + (\bar{c}_{13} + c_{55})u^{(0)}_{3,13} \\ &\quad + \kappa_1 c_{56} u^{(1)}_{1,3} + \kappa_3 \bar{c}_{14} u^{(1)}_{3,1} + \bar{e}_{11}\varphi^{(0)}_{,11} + e_{35}\varphi^{(0)}_{,33} + T^{(0)}_1 = \rho \ddot{u}^{(0)}_1, \end{aligned} \tag{26}$$

$$\begin{aligned} &(\kappa_1 c_{56} + \kappa_3 \bar{c}_{14})u^{(0)}_{1,13} + \kappa_1^2 c_{66} u^{(0)}_{2,11} + \kappa_3^2 \bar{c}_{44} u^{(0)}_{2,33} + \kappa_3 \bar{c}_{34} u^{(0)}_{3,33} \\ &\quad + \kappa_1^2 c_{66} u^{(1)}_{1,1} + \kappa_3^2 \bar{c}_{44} u^{(1)}_{3,3} + (\kappa_1 e_{36} + \kappa_3 \bar{e}_{14})\varphi^{(0)}_{,13} + T^{(0)}_2 = \rho \ddot{u}^{(0)}_2, \end{aligned} \tag{27}$$

$$\begin{aligned} &(\bar{c}_{13} + c_{55})u^{(0)}_{1,13} + \kappa_1 c_{56} u^{(0)}_{2,11} + \kappa_3 \bar{c}_{34} u^{(0)}_{2,33} + c_{55} u^{(0)}_{3,11} + \bar{c}_{33} u^{(0)}_{3,33} \\ &\quad + \kappa_1 c_{56} u^{(1)}_{1,1} + \kappa_3 \bar{c}_{34} u^{(1)}_{3,3} + (\bar{e}_{13} + e_{35})\varphi^{(0)}_{,13} + T^{(0)}_3 = \rho \ddot{u}^{(0)}_3, \end{aligned} \tag{28}$$

$$\begin{aligned} &\bar{e}_{11}u^{(0)}_{1,11} + e_{35}u^{(0)}_{1,35} + (\kappa_1 e_{36} + \kappa_3 \bar{e}_{14})u^{(0)}_{2,13} + (\bar{e}_{13} + e_{35})u^{(0)}_{3,13} \\ &\quad + \kappa_1 e_{36} u^{(1)}_{1,3} + \kappa_3 \bar{e}_{14} u^{(1)}_{3,1} - \varepsilon_{11}\varphi^{(0)}_{,11} - \varepsilon_{33}\varphi^{(0)}_{,33} + D^{(0)} = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &\gamma_{11}u^{(1)}_{1,11} + \gamma_{55}u^{(1)}_{1,33} + (\gamma_{13} + \gamma_{55})u^{(1)}_{3,13} + e^{(1)}_{11}\varphi^{(1)}_{,11} + e^{(1)}_{35}\varphi^{(1)}_{,33} \\ &\quad - 3b^{-2}[\kappa_1^2 c_{66}(u^{(0)}_{2,1} + u^{(1)}_1) + \kappa_1 c_{56}(u^{(0)}_{3,1} + u^{(0)}_{1,3}) + \kappa_1 e_{36}\varphi^{(0)}_{,3} + \kappa_1 e_{26}\varphi^{(1)}] + T^{(1)}_1 = \rho \ddot{u}^{(1)}_1, \end{aligned} \tag{30}$$

$$\begin{aligned} &(\gamma_{13} + \gamma_{55})u^{(1)}_{1,13} + \gamma_{55}u^{(1)}_{3,11} + \gamma_{33}u^{(1)}_{3,33} + (e^{(1)}_{13} + e^{(1)}_{35})\varphi^{(1)}_{,13} \\ &\quad - 3b^{-2}[\kappa_3^2 \bar{c}_{44}(u^{(0)}_{2,3} + u^{(1)}_3) + \kappa_3 \bar{c}_{14} u^{(0)}_{1,1} + \kappa_3 \bar{c}_{34} u^{(0)}_{3,3} + \kappa_3 \bar{e}_{14}\varphi^{(0)}_{,1}] + T^{(1)}_3 = \rho \ddot{u}^{(1)}_3, \end{aligned} \tag{31}$$

$$e^{(1)}_{11}u^{(1)}_{1,11}+e^{(1)}_{35}u^{(1)}_{1,33}+(e^{(1)}_{13}+e^{(1)}_{35})u^{(1)}_{3,13}-\varepsilon_{11}\varphi^{(1)}_{,11}-\varepsilon_{33}\varphi^{(1)}_{,33}$$
$$-3b^{-2}[\kappa_1 e_{26}(u^{(0)}_{2,1}+u^{(1)}_1)+e_{25}(u^{(0)}_{3,1}+u^{(0)}_{1,3})-\varepsilon_{23}\varphi^{(0)}_{,3}-\varepsilon_{22}\varphi^{(1)}]+D^{(1)}=0. \qquad (32)$$

In these equations,

$$\bar{c}_{11}=c_{11}-c_{12}^2/c_{22} \qquad \bar{c}_{13}=c_{13}-c_{12}c_{23}/c_{22}$$
$$\bar{c}_{33}=c_{33}-c_{23}^2/c_{22} \qquad \bar{c}_{14}=c_{14}-c_{12}c_{24}/c_{22}$$
$$\bar{c}_{44}=c_{44}-c_{24}^2/c_{22} \qquad \bar{c}_{34}=c_{34}-c_{23}c_{24}/c_{22}$$
$$\gamma_{11}=s_{33}/(s_{11}s_{33}-s_{13}^2) \qquad \gamma_{13}=-s_{13}/(s_{11}s_{33}-s_{13}^2)$$
$$\gamma_{33}=s_{11}/(s_{11}s_{33}-s_{13}^2) \qquad \gamma_{55}=1/s_{55}$$
$$\bar{e}_{11}=e_{11}-e_{12}c_{12}/c_{22} \qquad e^{(1)}_{11}=d_{11}\gamma_{11}+d_{13}\gamma_{13}$$
$$\bar{e}_{13}=e_{13}-e_{12}c_{23}/c_{22} \qquad e^{(1)}_{13}=d_{11}\gamma_{13}+d_{13}\gamma_{33}$$
$$\bar{e}_{14}=e_{14}-e_{12}c_{24}/c_{22} \qquad e^{(1)}_{35}=d_{35}\gamma_{55}.$$

Note that the primes employed by Sykes [8] are omitted here.

It remains only to fix the values of the shear coefficients κ_1 and κ_3. This is done [1] by equating the thickness-shear frequencies obtained from the two-dimensional equations (26)–(32) to the corresponding frequencies obtained from the three-dimensional equations.

In (26)–(32), set all spatial differential coefficients equal to zero. The last three equations then reduce to

$$-3b^{-2}(\kappa_1^2 c_{66}u^{(1)}_1+\kappa_1 e_{26}\varphi^{(1)})+T^{(1)}_1=\rho\ddot{u}^{(1)}_1, \qquad (33)$$

$$3b^{-2}\kappa_3^2\bar{c}_{44}u^{(1)}_3+T^{(1)}_3=\rho\ddot{u}^{(1)}_3, \qquad (34)$$

$$-3b^{-2}(\kappa_1 e_{26}u^{(1)}_1-\varepsilon_{22}\varphi^{(1)})+D^{(1)}=0. \qquad (35)$$

It may be seen that the thickness-shear displacement $u^{(1)}_1$ is coupled to the electric field, but $u^{(1)}_3$ is not. In the former case, a thickness-shear vibration may be driven by an alternating voltage $Ve^{i\omega t}$ applied to electrode films deposited on the faces of the plate: Then, if the two films are alike, the surface conditions are

$$T_{21}]_{\pm b}=\mp 2\rho' b'\ddot{u}_1]_{\pm b}, \qquad \varphi]_{\pm b}=\pm V\,e^{i\omega t},$$

where $2\rho' b'$ is the mass per unit area of each electrode film. Accordingly, in (33) and (35),

$$T^{(1)}_1=-3R\rho\ddot{u}^{(1)}_1, \qquad \varphi^{(1)}=V\,e^{i\omega t}/b,$$

where $R(=2\rho' b'/\rho b)$ is the ratio of the mass per unit area of both electrode films to the mass per unit area of the plate. If we set

$$u^{(1)}_1=A\,e^{i\omega t},$$

we find, from (33),

$$A=3\kappa_1 e_{26}V/b[b^2(1+3R)\rho\omega^2-3\kappa_1^2 c_{66}]. \qquad (36)$$

Hence, resonance occurs when

$$\omega^2 = 3\kappa_1^2 c_{66}/\rho b^2(1+3R), \qquad (\kappa_1 \text{ with electrodes}) \tag{37}$$

and the surface charge, off resonance, is obtained from $D^{(1)}$ in (35):

$$D_2]_{-b}^{b} = \tfrac{2}{3}b^2 D^{(1)} = 2b^{-1}(\kappa_1 e_{26} A - \varepsilon_{22} V)\,\mathrm{e}^{i\omega t},$$

where A is given in (36).

In the absence of electrode films, the traction and charge on the faces are zero, i.e.

$$T^{(1)} = 0, \qquad D^{(1)} = 0$$

in (33) and (35). Then with

$$u_1^{(1)} = A\,\mathrm{e}^{i\omega t}, \qquad \varphi^{(1)} = B\,\mathrm{e}^{i\omega t},$$

(33) and (35) become

$$(\kappa_1^2 c_{66} - \tfrac{1}{3}\rho b^2 \omega^2)A + \kappa_1 e_{26} B = 0,$$
$$\kappa_1 e_{26} A - \varepsilon_{22} B = 0,$$

from which, by elimination of A and B,

$$\omega^2 = 3\kappa_1^2 \bar{c}_{66}/\rho b^2 \quad (\kappa_1 \text{ without electrodes}) \tag{38}$$

where

$$\bar{c}_{66} = c_{66} + e_{26}^2/\varepsilon_{22}.$$

In the case of thickness-shear in the x_3-direction, (34) applies. When there are electrode films, we set

$$T_3^{(1)} = -3R\rho\ddot{u}_3^{(1)}, \qquad u_3^{(1)} = A\,\mathrm{e}^{i\omega t}$$

and find

$$\omega^2 = 3\kappa_3^2 \bar{c}_{44}/\rho b^2(1+3R) \quad (\kappa_3 \text{ with electrodes}). \tag{39}$$

In the absence of electrodes, $T_3^{(1)} = 0$ and (34) yields

$$\omega^2 = 3\kappa_3^2 \bar{c}_{44}/\rho b^2 \quad (\kappa_3 \text{ without electrodes}). \tag{40}$$

The four frequencies given in (37)–(40) are to be equated to the corresponding frequencies obtained from the three-dimensional equations (4) and boundary conditions (6) or (7).

For thickness-shear in the x_1-direction, the pertinent solutions of the three-dimensional equations are given by Bleustein and Tiersten [9]. For

$$R = 2\rho' b'/\rho b \ll 1, \qquad \kappa_{26}^2 = e_{26}^2/\varepsilon_{22}\bar{c}_{66} \ll 1,$$

they find, for the plate with electrodes,

$$\omega^2 = \pi^2 \bar{c}_{66}(1 - R - 4k_{26}^2/\pi^2)^2 4\rho b^2 \tag{41}$$

and, without electrodes,

$$\omega^2 = \pi^2 \bar{c}_{66}/4\rho b^2. \tag{42}$$

Hence, equating (41) and (42) to (37) and (38), respectively, we have, for $R \ll 1$, $k_{26}^2 \ll 1$,

$$\kappa_1^2 = (\pi^2/12)(1+R-8k_{26}^2/\pi^2) \quad \text{(with electrodes)}$$

$$\kappa_1^2 = \pi^2/12 \quad \text{(without electrodes).}$$

For thickness-shear in the x_3-direction, the solution of the three-dimensional equations is described in [10]. For $R \ll 1$, the frequency is found to be, after correcting an error in equation (43) of [10];

$$\omega^2 = \pi^2 c_3(1-R)^2/4\rho b^2, \tag{43}$$

where

$$c_3 = \tfrac{1}{2}\{c_{22}+c_{44}-[(c_{22}-c_{44})^2+4c_{24}^2]^{\frac{1}{2}}\}.$$

Without electrodes, the corresponding frequency is [11]

$$\omega^2 = \pi^2 c_3/4\rho b^2. \tag{44}$$

Hence, for $R \ll 1$, we have, upon equating (43) and (44) to (39) and (40), respectively,

$$\kappa_3^2 = (\pi^2/12)(1+R)c_3/\bar{c}_{44} \quad \text{(with electrodes)}$$

$$\kappa_3^2 = (\pi^2/12)c_3/\bar{c}_{44} \quad \text{(without electrodes).}$$

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(*Received* 6 *December* 1971)

Абстракт—Приводятся двухмерные уравнения движения пьезоэлектрических кристаллических пластинок, путём сохранения первых выражений в разложениях в степенный ряд, для механического перемещения и электрического потенциала, в вариационном принципе для трехмерных уравнений пьезоэлектричности.